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On the structure of non-commutative $N = 2$ Super Yang-Mills theory

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Abstract: We show that the recently proposed formulation of non-commutative $N = 2$ Super Yang-Mills theory implies that the commutative and non-commutative effective coupling constants $\tau(u)$ and $\tau_{nc}(u)$ coincide. We then introduce a key relation which allows to find a non-trivial solution of such equation, thus fixing the form of the low-energy effective action. The dependence on the non-commutative parameter arises from a rational function deforming the Seiberg-Witten differential.

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Non-commutative string and gauge theories have attracted strong attention [1, 2, 3]. It is well known that gauge theories on a non-commutative space-time can arise as the low-energy effective open string theory in the presence of D-branes with a non-vanishing NS-NS two-form B-field [2, 3, 4]. An interesting related investigation concerns the formulation of the non-commutative version of $N = 2$ Super Yang-Mills theory with gauge group U(2) [5, 6].

In this letter we argue that the deformation induced by the space-time noncommutativity can be neatly reabsorbed into a redefinition of the electric and magnetic masses a and a_D appearing in the BPS mass formula. In particular, we will derive an explicit expression for $a_{D,nc}$ and a_{nc} which denote the non-commutative analogues of a_D and a.

In [6] it has been found that, under reasonable assumptions, $a_{D,nc}$ and a_{nc} have the same monodromies as their commutative partners [7]. Furthermore, the same elliptic curve that first appeared in [7] has been found to describe the non-commutative theory. The asymptotic behavior at $u = \infty$ is the same as in the commutative Seiberg-Witten model, i.e.

$$a_{D,nc}(u \longrightarrow \infty) \sim \frac{i}{\pi}\sqrt{2u}\ln\frac{u}{\Lambda^2}\,, \qquad a_{nc}(u \longrightarrow \infty) \sim \sqrt{2u}\,. \tag{1}$$

However, the asymptotic behavior of a and a_D in the dual U(1) phase differs from its commutative counterpart, since the β-function gets also a contribution from the U(1) gauge multiplet, which renders this theory asymptotically free [8]. In fact, at $u = \Lambda^2$ we have

$$a_{D,nc}(u \longrightarrow \Lambda^2) \sim c_0(u - \Lambda^2)^{-1}\,, \tag{2}$$

which has to be compared with the commutative case

$$a_D(u \longrightarrow \Lambda^2) \sim \frac{i}{2\Lambda}(u - \Lambda^2)\,. \tag{3}$$

Following these assumptions, in this letter we propose a definition of a_{nc} and $a_{D,nc}$ through a simple modification of the Seiberg-Witten differential, and therefore of a and a_D, which provides them with the same monodromies and asymptotic properties of a_{nc} and $a_{D,nc}$.

The framework of the derivation is similar to the one used in [9] to prove the uniqueness of the Seiberg-Witten solution by means of reflection symmetry of the quantum vacua.

According to [6], the behavior of the non-commutative effective gauge coupling constant τ_{nc} (as a function of u) for $u \to \infty$ and $u = +\Lambda^2$ is the same of τ. Furthermore, since a_{nc} and $a_{D,nc}$ have the same monodromy of a_D and a, it follows that τ_{nc} has the same monodromy of τ. A further physical requirement on τ_{nc} is the positivity of its imaginary part

$$\mathrm{Im}\,\tau_{nc} = \frac{4\pi}{g^2} > 0\,. \tag{4}$$

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On the other hand, we know that the u moduli space is the thrice punctured Riemann sphere. Thus, on general grounds, we can use the standard arguments of the uniformization theory, concerning the uniqueness of the uniformizing map [10, 9], to see that

$$\tau_{nc}(u) = \tau(u) \,. \tag{5}$$

This is a key point since it will lead us to fix the (polymorphic) functions a_{nc} and $a_{D,nc}$. Actually, we will present an argument, which is in fact of interest also in uniformization theory, which will lead us to find a non-trivial solution to the following question. While on one side we have $\tau_{nc}(u) = \tau(u)$, on the other side we have that a_{nc} and $a_{D,nc}$ do not coincide with a and a_D. Thus we are led to formulate the following problem:

Given two sets of polymorphic functions $(a_{D,nc}, a_{nc})$ and (a_D, a), having the same monodromy transformations, find non-trivial solutions of eq. (5), that is

$$\frac{\partial_u a_{D,nc}}{\partial_u a_{nc}} = \frac{\partial_u a_D}{\partial_u a} \,. \tag{6}$$

Since a_{nc} and $a_{D,nc}$ have the same monodromies as a and a_D, it would seem at first sight that $(a_{D,nc}, a_{nc}) = h(u)(a_D, a)$, where h is a function of u with trivial monodromies. However, this would not solve eq. (6), unless $h = cnst$. Since from (2) and (3) we have $(a_{D,nc}, a_{nc}) \not\propto (a_D, a)$, it is clear that we have to look for other functions. This is an important point because the proposal in [6] may be implemented only if (6) admits non-trivial solutions. It is remarkable that these solutions indeed exist. Let us start by recalling the differential equation [11, 10]

$$\left(\partial_u^2 + \frac{1}{4(u^2 - \Lambda^4)}\right) \begin{pmatrix} a_D \\ a \end{pmatrix} = 0 \,. \tag{7}$$

We then consider two functions $f(u)$ and $g(u)$ with trivial monodromy around $u = \infty$, $u = \pm\Lambda^2$, and set

$$a_{D,nc} = f(u) a_D + g(u) a'_D \,, \qquad a_{nc} = f(u) a + g(u) a' \,, \tag{8}$$

where $' \equiv \partial_u$. Note that $(a_{D,nc}, a_{nc})$ in (8) have the same monodromy of (a_D, a), i.e.

$$\begin{pmatrix} a_{D,nc} \\ a_{nc} \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{a}_{D,nc} \\ \tilde{a}_{nc} \end{pmatrix} = M \begin{pmatrix} a_{D,nc} \\ a_{nc} \end{pmatrix} , \tag{9}$$

where

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} , \qquad M_{+1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} , \qquad M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} , \tag{10}$$

are the monodromies around $u = \infty, +\Lambda^2$ and $-\Lambda^2$, respectively.

The crucial observation is that $a'_{D,nc}$ and a'_{nc} still have the same form of (8) with new functions $\tilde{f}$ and $\tilde{g}$. Actually, from (7) and (8) we have

$$a'_{D,nc} = \tilde{f}(u)a_D + \tilde{g}(u)a'_D\,, \qquad a'_{nc} = \tilde{f}(u)a + \tilde{g}(u)a'\,, \tag{11}$$

where

$$\tilde{f}(u) = f'(u) - \frac{1}{4(u^2-\Lambda^4)}g(u)\,, \qquad \tilde{g}(u) = f(u) + g'(u)\,. \tag{12}$$

It is now clear what the form of the solutions of eq. (6) is. In fact, requiring $\tilde{f} = 0$, that is

$$f'(u) - \frac{1}{4(u^2-\Lambda^4)}g(u) = 0\,, \tag{13}$$

we get the key relation

$$a'_{D,nc} = H(u)a'_D\,, \qquad a'_{nc} = H(u)a'\,, \tag{14}$$

where

$$H(u) = f + 8uf' + 4(u^2-\Lambda^4)f''\,. \tag{15}$$

Summarizing, from (8) and (13) we have

$$a_{D,nc} = f(u)a_D + 4(u^2-\Lambda^4)f'(u)a'_D\,, \qquad a_{nc} = f(u)a + 4(u^2-\Lambda^4)f'(u)a'\,. \tag{16}$$

which satisfy (6) since, from (14) we see that

$$\tau_{nc} = \frac{a'_{D,nc}}{a'_{nc}} = \frac{H(u)a'_D}{H(u)a'} = \tau\,. \tag{17}$$

Until now we have derived a set of solutions of eq. (6) depending on the function f. Comparing (1), (2) and (3) with (16), we see that the function f should satisfy the conditions

$$f(u \longrightarrow \infty) = 1\,, \qquad f(u \longrightarrow \Lambda^2) \sim d_0(u-\Lambda^2)^{-2}\,. \tag{18}$$

Let us set

$$f(u) = \frac{P(u)}{Q(u)}\,. \tag{19}$$

P and Q should be polynomial functions, since otherwise we would get singularities not found in the asymptotic analysis. The first condition in (18) fixes P and Q to be of the same degree, while from the second condition we obtain

$$Q(u) = (u-\Lambda^2)^2 \sum_{k=0}^{N} c_k u^k\,. \tag{20}$$

Due to the singularity structure, it is reasonable to assume that the only possible poles in the finite region of the moduli space arise at the punctures $u = \pm\Lambda^2$. Another condition concerns the $\mathbb{Z}_2$ symmetry of the moduli space. To understand this, let us recall that, in the commutative case, $a_D(e^{i\pi/2}a) = a_D - a$ and $a(-u) = e^{i\pi/2}a$ [9]. In order to preserve these properties for $a_{D,nc}$ and a_{nc}, we need that $P(-u) = P(u)$ and $Q(-u) = Q(u)$, so that

$$Q(u) = (u^2 - \Lambda^4)^2 \sum_{k=0}^{N-2} \tilde{c}_k u^k \,. \tag{21}$$

Thus we end with an expression which is singular at $u = \pm\Lambda^2$. Concerning the coefficients $\tilde{c}_k$ we note that $\tilde{c}_{k\neq 0} = 0$, since otherwise we would have poles outside the critical points.

Summarizing, we have

$$f(u) = \frac{u^4 + \alpha u^2 + \beta}{(u^2 - \Lambda^4)^2} \,, \tag{22}$$

where α and β are functions of Λ and of the non-commutative parameter θ. Note that this implies that the constants c_0 and d_0 in (2) and (18), are

$$c_0 = \frac{i}{2\Lambda} d_0 \,, \qquad d_0 = \Lambda^8 + \alpha\Lambda^4 + \beta \,. \tag{23}$$

There is still one more condition we have to satisfy. Namely, in the $\theta \to 0$ limit, $(a_{D,nc}, a_{nc})$ should reduce to (a_D, a). This implies that $\lim_{\theta\to 0} f = 1$, that is

$$\lim_{\theta\longrightarrow 0} \alpha = -2\Lambda^4 \,, \qquad \lim_{\theta\longrightarrow 0} \beta = \Lambda^8 \,. \tag{24}$$

These conditions together with dimensional analysis imply

$$\alpha = \Lambda^4 \left[-2 + \sum_{k=1}^{\infty} \alpha_k (\theta\Lambda^2)^k \right] , \qquad \beta = \Lambda^8 \left[1 + \sum_{k=1}^{\infty} \beta_k (\theta\Lambda^2)^k \right] . \tag{25}$$

Notice that the expressions of a and a_D get modified to

$$a_{D,nc} = 2 \int_{\Lambda^2}^{u} \lambda_{nc} \,, \qquad a_{nc} = 2 \int_{-\Lambda^2}^{\Lambda^2} \lambda_{nc} \,, \tag{26}$$

where, from (16)

$$\lambda_{nc} = f\lambda + 4(u^2 - \Lambda^4) f' \lambda' \,, \tag{27}$$

where λ stands for the Seiberg-Witten differential

$$\lambda = \frac{\sqrt{2}}{2\pi} \frac{dx\sqrt{x - u}}{\sqrt{x^2 - \Lambda^4}} \,, \tag{28}$$

Besides the divergence in the mass of the monopole found in [6], we see that the BPS mass formula has divergences both at $u = \Lambda^2$ and $u = -\Lambda^2$ for any non-trivial value of n_e and n_m

$$M = \sqrt{2} |n_e a_{nc} + n_m a_{D,nc}| \,. \tag{29}$$

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It is of great importance to investigate the structure of the expansions for α and β. Their explicit form will determine the critical values of θ, n_e and n_m corresponding to possible cancellations of divergences and the appearance of possible zeros for M. Let us conclude by observing that, despite many technical difficulties, a non-commutative analogue [12] of the analysis of the instanton calculations performed in the context of the standard Seiberg-Witten model [13, 14] is relevant in order to fix the structure of α and β.

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